(1.5)

Vectors in n-space

In previous sections, we have studied how the geometric properties of 2 and 3 dimensional Euclidean spaces can be represented algebraically by means of vectors.
Although they are considerably less intuitive, higher-dimensional spaces have many obvious applications.
For example, if you have ever made Borscht, you have encountered an 11-dimensional space:

Borscht Recipe

1. 8 cups beef broth
2. 1 pound slice of meaty bone-in beef shank
3. 1 large onion, peeled, quartered
4. 4 large beets
5. 4 carrots
6. 1 large russet potatoe
7. 2 cups thinly sliced cabbage
8. $\frac{3}{4}$ cup chopped fresh dill
9. 3 tbsp red wine vinegar
10. 1 cup sour cream
11. 6 tbsp of salt.

This ingredient list can be written as the vector $\vec{b} =$
$= (8, 1, 1, 4, 4, 1, 2, \frac{3}{4}, 3, 1, 6) \in \mathbb{R}^{11}$

More generally $\vec{x} = (x_1, \ldots, x_n)$ with $x_1, \ldots, x_n \in \mathbb{R}$ and n being a positive integer might be interpreted as a recipy with n ingredients* This vector $\vec{x}$ is said to belong to the set of all ordered n-tuples $\mathbb{R}^n$.

If $\vec{x}$ gives enough ingredients to make Borscht for 6 people, what would be the coordinates of the vector $\vec{y}$, which lists the amount of ingredients required to feed 12 people? Well, it seems that we must double the amount of each ingredient getting $\vec{y} = (2x_1, \ldots, 2x_n)$. On the other hand, this would be the same as using $\vec{x}$ twice (i.e. cooking one 6-man serving and then another 6-man serving), or $2\vec{x}$. Thus we get

$\vec{y} = (2x_1, \ldots, 2x_n) = 2\vec{x} = 2(x_1, \ldots, x_n)$

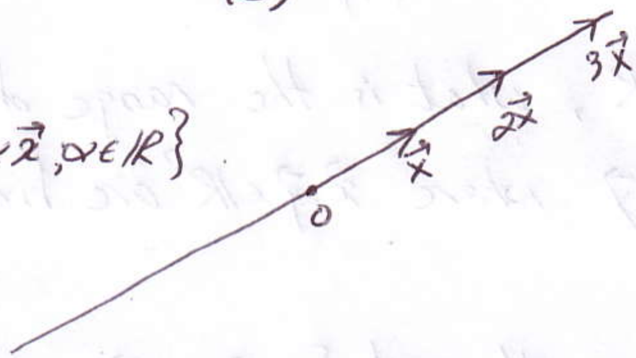
More generally, if you wish to cook for 6α people, $\alpha \in \mathbb{R}$, you must multiply each ingredient by α. In other words, your recipy will have to be $\alpha\vec{x} = \alpha(x_1, x_2, \ldots, x_n) =$

$= (\alpha x_1, \alpha x_2, \ldots, \alpha x_n)$.

Note that if you follow recipy $\vec{x}$ or recipy $\alpha\vec{x}$, you will get the same tasting soup. The thing that varies is the quantity of soup produced. If we think of the function $h: \mathbb{R} \to \mathbb{R}^n$, defined by $h(\alpha) = \alpha\vec{x}$ in geometric terms, then the range of h determines a line.

* Assuming that these ingredients are fixed.

$L = \{\vec{y} : \vec{y} = \alpha\vec{x}, \alpha \in \mathbb{R}\}$



As far as our taste buds are concerned, the same L is the collection of all reciples $\vec{y}$ that produce same-tasting soup. Any vector $\vec{y}$ that 'produces' the same-tasting soup as that of $\vec{x}$ can conversely be represented as $\alpha\vec{x}$ for some α.

In other words, two recipies produce the same soup taste iff they are linearly dependent.

Ex. Suppose $\vec{x} = (8, 1, 1, 4, 1, 2, 3/4, 3, 1, 6)$ as before.
Let $\vec{y} = (8, 1, 1, 4, 1, 2, 3/4, 3, 2, 6)$. Are $\vec{x}$ and $\vec{y}$ linearly dependent? Justify your answer both intuitevely and mathematically.

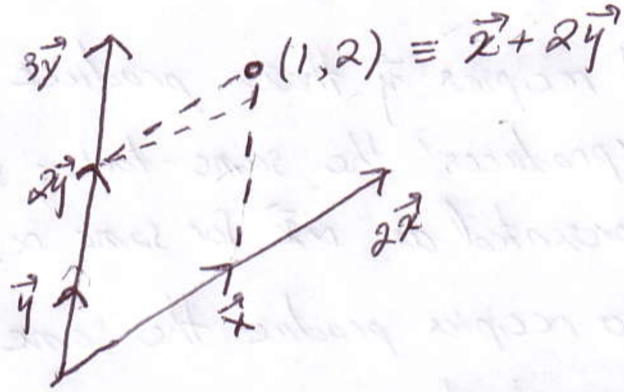
If $\vec{x}, \vec{y} \in \mathbb{R}^n$, how might we interpret $\vec{x} + \vec{y}$? If $\vec{x}, \vec{y} \in \mathbb{R}^n$ they might be thought of as variants of the Borscht recipe.
$\vec{x} + \vec{y}$ would then be the soup that results from mixing the Borscht cook according to recipe $\vec{x}$ with the Borscht cooked according to recipe $\vec{y}$. This soup will taste just like the Borscht whose recipe combines the ingredients of $\vec{x}$ and $\vec{y}$: if $\vec{x} = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$

$\vec{x} + \vec{y} = (x_1 + y_1, \ldots, x_n + y_n)$

Ex. If $\alpha, \beta \in \mathbb{R}$, what is the range of the function $f(\alpha, \beta) = \alpha \vec{x} + \beta \vec{y}$ where $\vec{x}, \vec{y} \in \mathbb{R}^n$ are fixed.

Solution:

The range of f is the set $\{\alpha \vec{x} + \beta \vec{y} ; \alpha, \beta \in \mathbb{R}\}$ which is a plane.



This plane is a 2-D space in an n-D space, because it generates a coordinate grid $(\alpha, \beta) \in \mathbb{R}^2$. Indeed, you may think of f as a function that identifies a point in the plane by means of its coordinates (α, β).
If $\vec{x}$ and $\vec{y}$ are linearly independent, the plane spanned by these vectors corresponds to the set of recipes that can be generated from two "completely distinct" ones. Now you know what makes some soups taste a little plane!

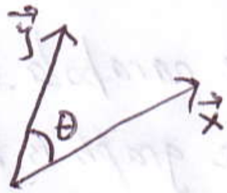
Generalizing $\vec{i}, \vec{j}, \vec{k}$

Any vector $\vec{x} = (x_1, \dots, x_n) = x_1(1, 0, \dots, 0) + x_2(0, 1, \dots, 0) + x_n(0, 0, \dots, 1)$ $= x_1\vec{e}_1 + x_2\vec{e}_2 + \dots + x_n\vec{e}_n$ where $\vec{e}_i = (0, \dots, 1, \dots 0)$ has 1 in the ith place and 0 everywhere else.

Ex. the vector $(5,3,-1) = 5\vec{i} + 3\vec{j} - \vec{k} = 5\vec{e}_1 + 3\vec{e}_2 - \vec{e}_3$ and the vector $(-2,7,5,-3) \in \mathbb{R}^4$ can be represented as $-2\vec{e}_1 + 7\vec{e}_2 + 5\vec{e}_3 - 3\vec{e}_4$

Generalizing the dot (inner) product

Two vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$ may be thought as lying in a 2-D plane. By reasons similar to our motivation of 1.2 we define $\vec{x} \cdot \vec{y} = \|\vec{x}\| \|\vec{y}\| \cos\theta$ where $\|\vec{x}\| = \sqrt{x_1^2 + \dots + x_n^2}$ and $\|\vec{y}\| = \sqrt{y_1^2 + \dots + y_n^2}$ and θ is the angle in the plane spanned by $\vec{x}$ and $\vec{y}$



It can be shown that $\vec{x} \cdot \vec{y} = x_1 y_1 + \dots + x_n y_n$

Note that length of $\vec{x} = \|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}}$.

Just as before, the angle θ may be obtained from the formula

$$\theta = \cos^{-1} \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \|\vec{y}\|}$$

You should verify the following properties of the dot product:

(i) $(\alpha\vec{x} + \beta\vec{y}) \cdot \vec{z} = \alpha(\vec{x} \cdot \vec{z}) + \beta(\vec{y} \cdot \vec{z})$

(ii) $\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$

(iii) $\vec{x} \cdot \vec{x} \geq 0$ with equality iff $\vec{x} = \vec{0}$.

Just like in $\mathbb{R}^2$ and $\mathbb{R}^3$, there are important inequalities, which you should know.

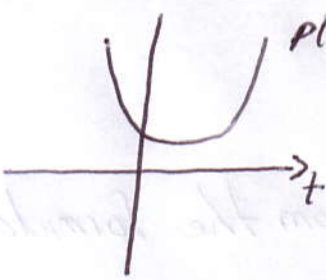
Cauchy - Schwarz Inequality: Let $\vec{x}, \vec{y} \in \mathbb{R}^n$ then $|\vec{x}\cdot\vec{y}| \leq \|\vec{x}\|\,\|\vec{y}\|$.

Proof: This theorem is so important that we'll do well to present a different proof (one that generalizes to non-Euclidian spaces) to the one given in section 1.2.

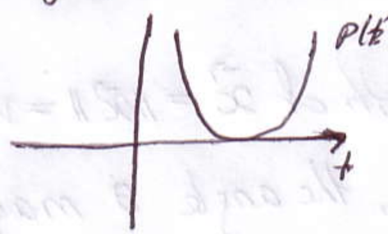
Notice that for $t \in \mathbb{R}$ $0 \leq \|t\vec{x}+\vec{y}\|^2 = (t\vec{x}+\vec{y})(t\vec{x}+\vec{y}) =$

$$= t^2(\vec{x}\cdot\vec{x}) + 2t(\vec{x}\cdot\vec{y}) + (\vec{y}\cdot\vec{y}) = t^2\|\vec{x}\|^2 + 2t(\vec{x}\cdot\vec{y}) + \|\vec{y}\|^2 \qquad (1)$$

But this means that $\|t\vec{x}+\vec{y}\|^2$ is a parabola in t with equation $p(t) = t^2\|\vec{x}\|^2 + 2t(\vec{x}\cdot\vec{y}) + \|\vec{y}\|^2$, whose graph is above the t-axis:



or



The minimum of p occurs at that value of t, for which $p'(t) = 0$ or $2t\|\vec{x}\|^2 + 2(\vec{x}\cdot\vec{y}) = 0$. Thus $t = \dfrac{-(\vec{x}\cdot\vec{y})}{\|\vec{x}\|^2}$

For this value of t, equation (1) becomes

$$0 \leq \frac{(\vec{x}\cdot\vec{y})^2}{\|\vec{x}\|^2} + \frac{-2(\vec{x}\cdot\vec{y})^2}{\|\vec{x}\|^2} + \|\vec{y}\|^2 \qquad (2)$$

or

$$\frac{(\vec{x}\cdot\vec{y})^2}{\|\vec{x}\|^2} \leq \|\vec{y}\|^2$$ upon multiplying both sides by $\|\vec{x}\|^2$

this inequality reduces to the desired result

$$(\vec{x}\cdot\vec{y})^2 \leq \|\vec{x}\|^2\|\vec{y}\|^2 \implies |\vec{x}\cdot\vec{y}| \leq \|\vec{x}\|\,\|\vec{y}\|$$

Corolary: Triangle inequality in $\mathbb{R}^n$.

Let $\vec{x},\vec{y}\in\mathbb{R}^n$. Then $\|\vec{x}+\vec{y}\| \leq \|\vec{x}\|+\|\vec{y}\|$.

Proof: $\|\vec{x}+\vec{y}\|^2 = (\vec{x}+\vec{y})\cdot(\vec{x}+\vec{y}) = \vec{x}\cdot\vec{x} + 2\vec{x}\cdot\vec{y} + \vec{y}\cdot\vec{y} =$

$= \|\vec{x}\|^2 + 2(\vec{x}\cdot\vec{y}) + \|\vec{y}\|^2 \leq \|\vec{x}\|^2 + 2|\vec{x}\cdot\vec{y}| + \|\vec{y}\|^2 \leq \|\vec{x}\|^2 + 2\|\vec{x}\|\,\|\vec{y}\| + \|\vec{y}\|^2$

$= (\|\vec{x}\|+\|\vec{y}\|)^2$

Hence $\|\vec{x}+\vec{y}\| \leq \|\vec{x}\|+\|\vec{y}\|$.

Distance between two vectors

If $\vec{x},\vec{y}\in\mathbb{R}^n$ then $d(\vec{x},\vec{y}) = \|\vec{x}-\vec{y}\|$.

General Matrices

Def: An $m\times n$ matrix A is an array of mn numbers:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

This matrix might sometimes be written, for brevity, as $[a_{ij}]$.

If A, B are two $m \times n$ matrices, we define $A+B$ componentwise. That is

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & \cdots & a_{1n}+b_{1n} \\ \vdots & & \vdots \\ a_{m1}+b_{m1} & \cdots & a_{mn}+b_{mn} \end{pmatrix}$$

Also if $\alpha \in \mathbb{R}$, we define the matrix αA by

$$\alpha A = \alpha \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} \alpha a_{11} & \cdots & \alpha a_{1n} \\ \vdots & & \vdots \\ \alpha a_{m1} & \cdots & \alpha a_{mn} \end{pmatrix}$$

Ex,

(a) $$\begin{pmatrix} 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 1 & 3 \\ 0 & 0 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 8 \end{pmatrix}$$

(b) $(1\ \ 2) + (0, -1) = (1, 1)$

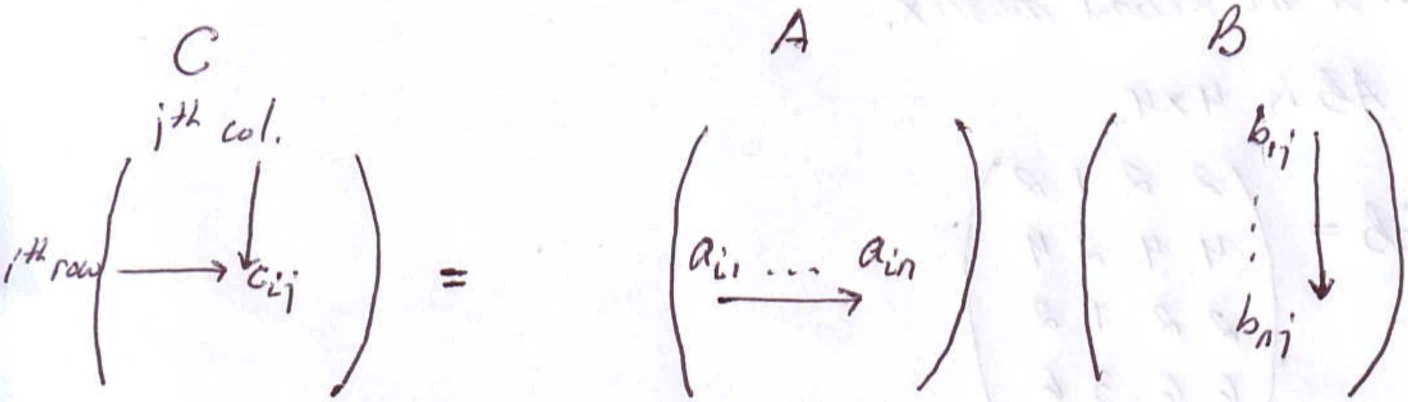
(c) $$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

(d) $$-3 \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 5 \\ 1 & 0 & 3 \end{pmatrix} = \begin{pmatrix} -3 & 3 & -6 \\ 0 & -3 & -15 \\ -3 & 0 & -9 \end{pmatrix}$$

Another useful matrix operation is called matrix multiplication. Unfortunately, matrix multiplication is not quite as easy as the previous operations.

Def: let A be an $m \times n$ matrix and let B be an $n \times p$ matrix. Then AB is the $m \times p$ matrix C with entries $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$.

In other words, c_{ij} is obtained by taking the dot product of the i^{th} row of A with the j^{th} column of B.



Please note that matrix multiplication is only defined if the left matrix A has the same number of columns as the number of rows of B.

Ex. Let $A = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Then

A is an 2×3 matrix and B is a 3×3 matrix

Hence $AB = \begin{pmatrix} 3 & 1 & 5 \\ 3 & 4 & 5 \end{pmatrix}$ what can you say about BA?

Ex. let $A = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix}$ and $B = (2\ 2\ 1\ 2)$

Then A is 4×1 and B is 1×4

AB is defined because it is the product of a matrix A, that has 1 column, and matrix B that has one row. In other words the first matrix has the same number of columns as the number of rows of the second matrix.

Hence AB is 4×4.

$$AB = \begin{pmatrix} 2 & 2 & 1 & 2 \\ 4 & 4 & 2 & 4 \\ 2 & 2 & 1 & 2 \\ 6 & 6 & 3 & 6 \end{pmatrix}$$

Similarly BA is defined

$BA = (13)$ Obviously $AB \neq BA$

Matrix product is generally not commutative even when the matrices are of the same size:

Ex. let

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

then $AB = \begin{pmatrix} 0 & 4 & 3 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $BA = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 3 \\ 3 & 1 & 0 \end{pmatrix}$

Observe that $AB \neq BA$

<u>The transpose of a matrix</u>

Def: Given $m \times n$ matrix $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$, we define

the $n \times m$ matrix $A^T = \begin{pmatrix} a_{11}^T & \cdots & a_{1m}^T \\ \vdots & & \vdots \\ a_{n1}^T & \cdots & a_{nm}^T \end{pmatrix}$, called the transpose

of A, by $(a_{ij}^T) = (a_{ji})$. That is, A^T is obtained from A by making the first column of A the first row of A^T, the second column of A – the second row of A^T and so on. In other words,

$$A^T = \begin{pmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & & \vdots \\ a_{1n} & \cdots & a_{nm} \end{pmatrix}$$

Ex. Let $A = \begin{pmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{pmatrix}$ then $A^T = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{pmatrix}$

Comp. check: Compute the transpose

(a) $\begin{pmatrix} 1 & -3 & 4 \\ 2 & 2 & -10 \\ 4 & 0 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & -7 & 4 & 5 \end{pmatrix}$ (c) $\begin{pmatrix} 0 \\ -3 \\ 4 \\ 6 \end{pmatrix}$

(d) (13)